Newton polygons for the non-bipartite dimer model

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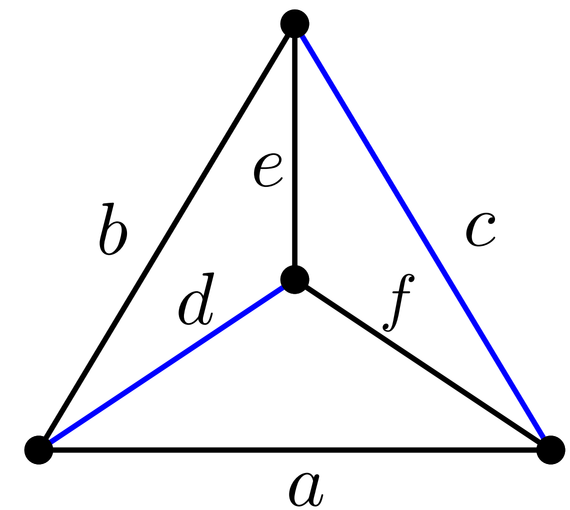
Dimer model

- Planar graphs with positive edge weights.

- A *dimer cover* is a subset of edges such that each vertex is covered by exactly one edge.



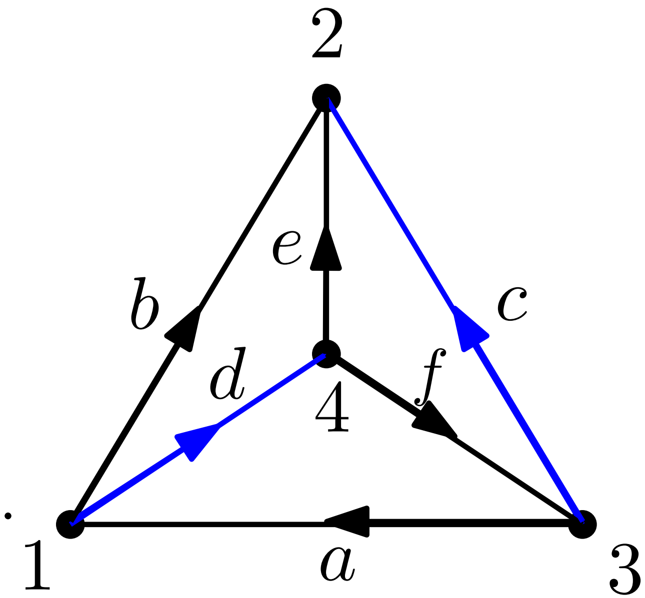
dimer cover with weight cd

- Boltzmann probability measure: draw a dimer cover at random with probability proportional to its weight.

- Partition function and correlations were exactly computed in 1960s (Kasteleyn, Temperley, Fisher)

Kasteleyn matrix

$$K = \begin{bmatrix} 0 & b & -a & d \\ -b & 0 & -c & -e \\ a & c & 0 & -f \\ -d & e & f & 0 \end{bmatrix}$$



Kasteleyn signs respect the clockwise-odd rule.

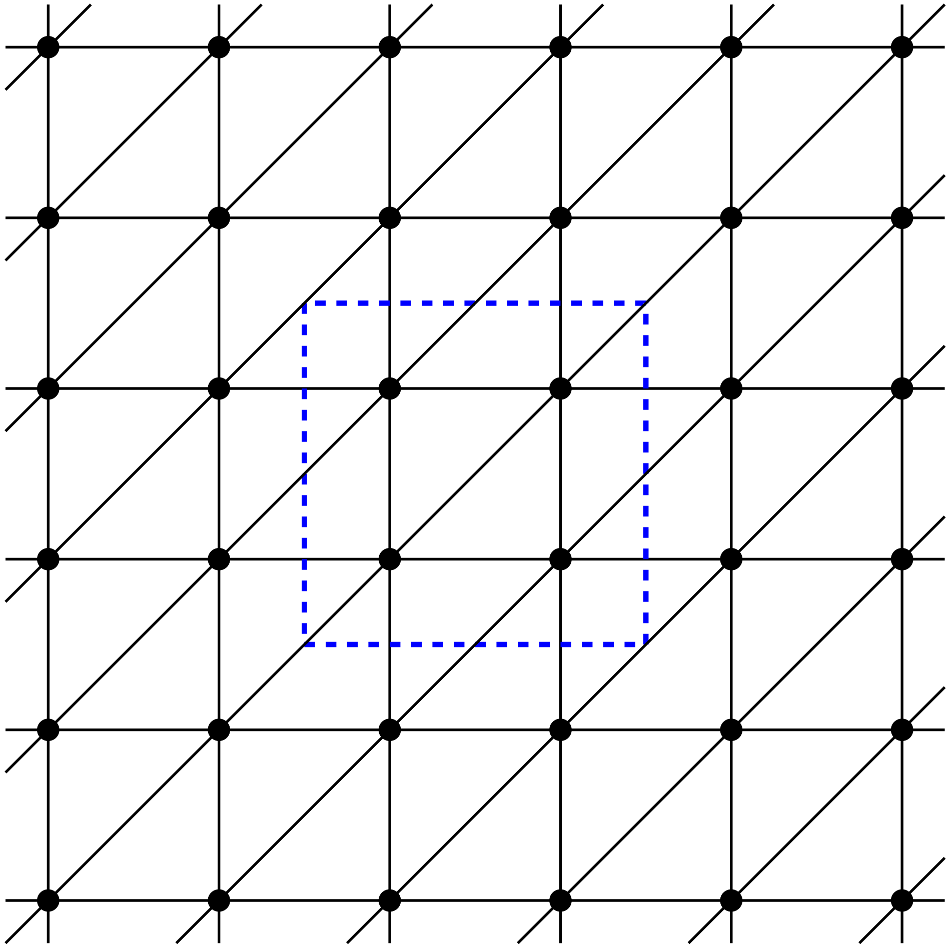
$$\det K = a^2e^2 + b^2f^2 + c^2d^2 + 2abef + 2acde + 2bcdf = (ae + bf + cd)^2$$

Boltzmann measure: $\mathbb{P}(M) = \frac{wt(M)}{Z}$

$Z = \sqrt{\det K}$ denotes the partition function.

$$\mathbb{P}(M) = \frac{cd}{ae+bf+cd}$$

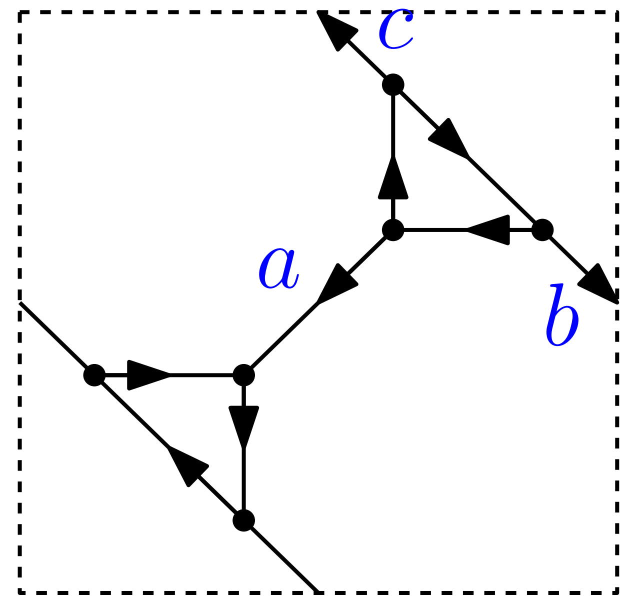
Characteristic polynomial



There is a simple way to associate to each infinite graph Γ periodic in two directions a Laurent polynomial $P(z, w)$, which is called the characteristic polynomial associated to the dimer model of Γ.

The modified Kasteleyn matrix

$$K(z,w) = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & cw \\ -1 & 0 & 1 & bz & 0 & 0 \\ 1 & -1 & 0 & 0 & a & 0 \\ 0 & -\frac{b}{z} & 0 & 0 & 1 & -1 \\ 0 & 0 & -a & -1 & 0 & 1 \\ -\frac{c}{w} & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$



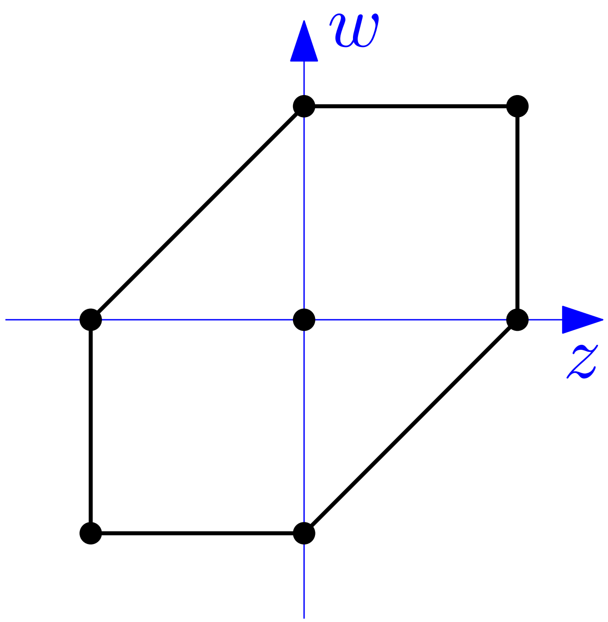
$$P(z,w) = \det(K(z,w)) = a^2 + b^2 + c^2 + a^2b^2c^2 + ab(1-c^2)(z+1/z)$$
$$+ac(1-b^2)(w+1/w) + bc(1-a^2)(w/z + z/w)$$

The *spectral curve* is the set of all $(z,w) \in (\mathbb{C}^*)^2$ such that $P(z,w) = 0$.

Newton polygon

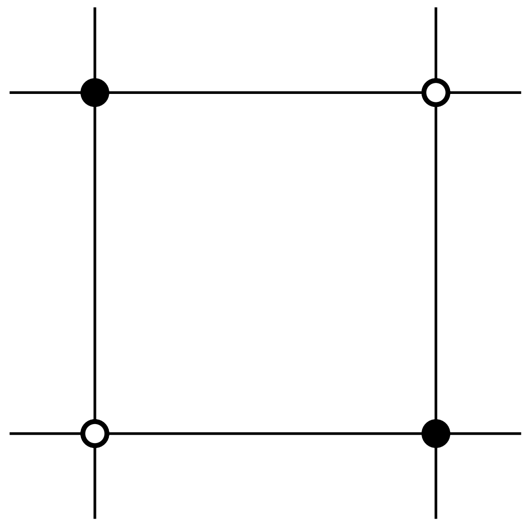
The Newton polygon of $P(z,w)$ is the convex hull of all the $(i,j) \in \mathbb{Z}^2$, such that $P(z,w)$ has a monomial of the form $a_{i,j} z^i w^j$.

$$P(z,w) = a_{0,0} + a_{1,0} z + a_{0,1} w + a_{1,1} zw + \frac{a_{-1,0}}{z} + \frac{a_{0,-1}}{w} + \frac{a_{-1,-1}}{zw}.$$



Bipartite dimer model

- We say that a graph is *bipartite* if we can divide its vertices into black and white vertices, such that each edge connects one black and one white vertex.



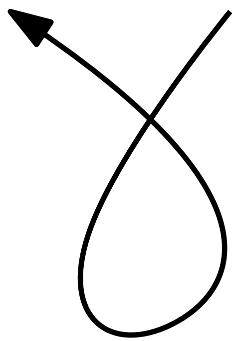
- Kenyon, Okounkov, and Sheffield (2006) described the characteristic polynomials arising from bipartite dimer models.

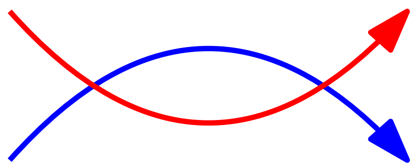
- Goncharov and Kenyon (2013) introduced an integrable system from bipartite dimer models associated with Newton polygons.

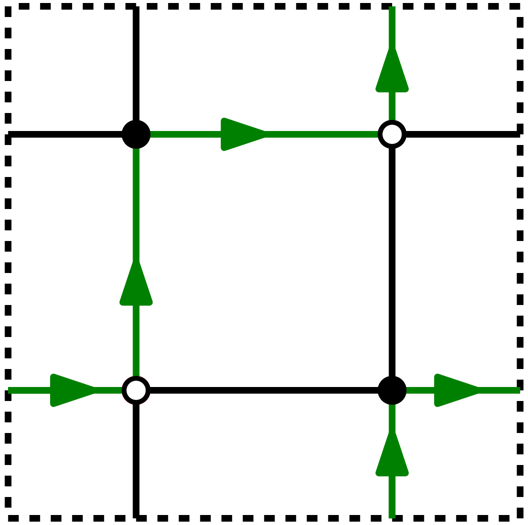
Zig-zag paths

A *zig-zag* path on a bipartite graph Γ is an oriented path that turns maximally left on white vertices and maximally right on black vertices.

A bipartite graph Γ is *minimal*, if it contains no self-intersecting zig-zag paths and no two zig-zag paths intersect twice with the same orientation.

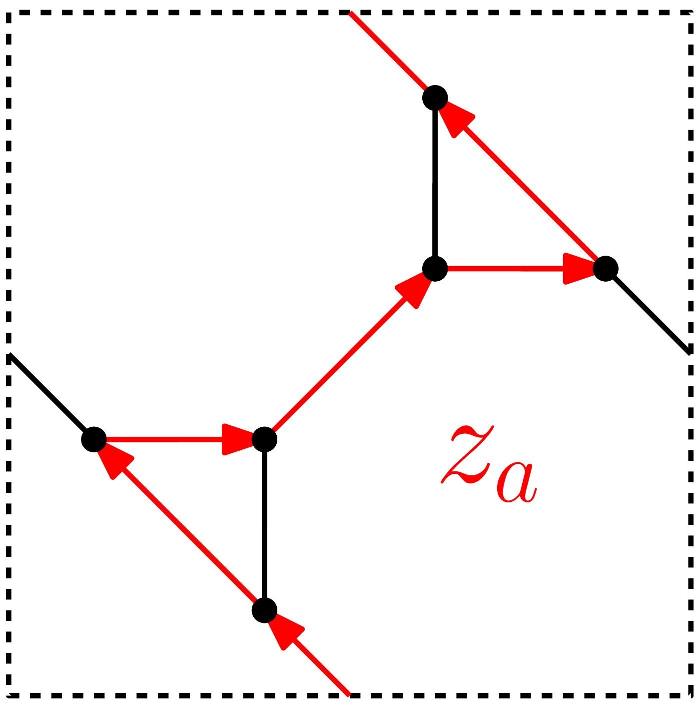






Homology of zig-zag paths

For non-bipartite graphs, *zig-zag* paths are oriented paths that alternately turn maximally left and maximally right.



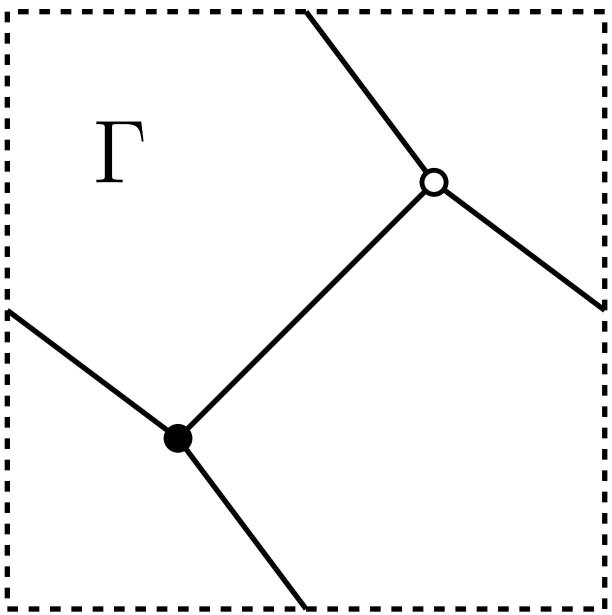
The homology class $(a, b) \in \mathbb{Z}^2$ of a zig-zag path encodes the number of times it winds horizontally (resp. vertically) around the torus.

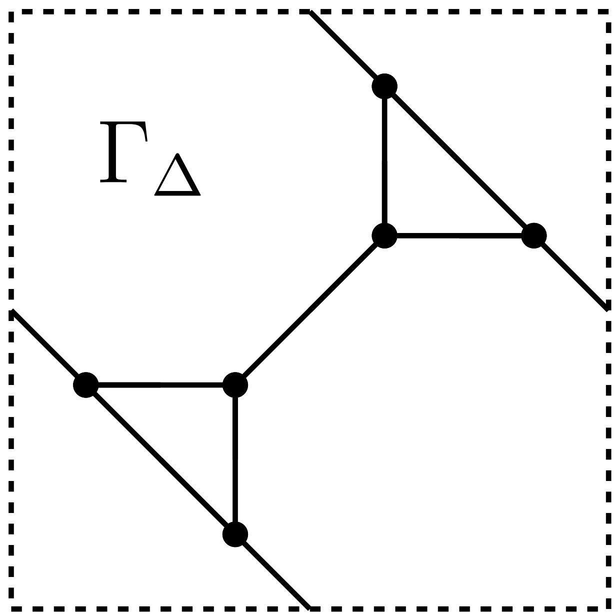
The zig-zag path z_a has the homology class $(0, 1)$.

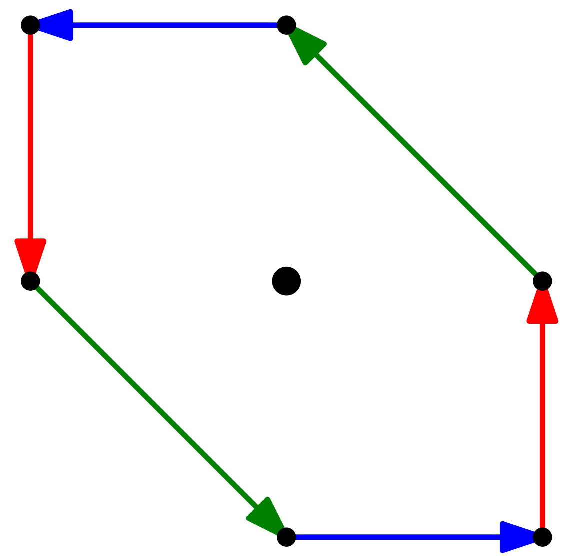
Theorem 1 [B. 2025]

Let Γ be a minimal bipartite graph with vertices of degree 2 or 3, and Γ_Δ the graph obtained by replacing each degree 3 vertex in Γ with a triangle. Then the Newton polygons of Γ and Γ_Δ are the same.

Furthermore, the edge vectors of the Newton polygon of Γ_Δ are obtained by homology classes of its zig-zag paths.

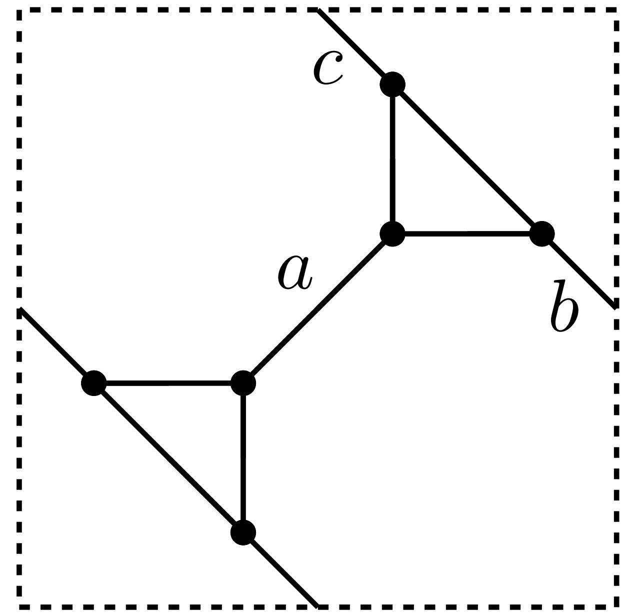






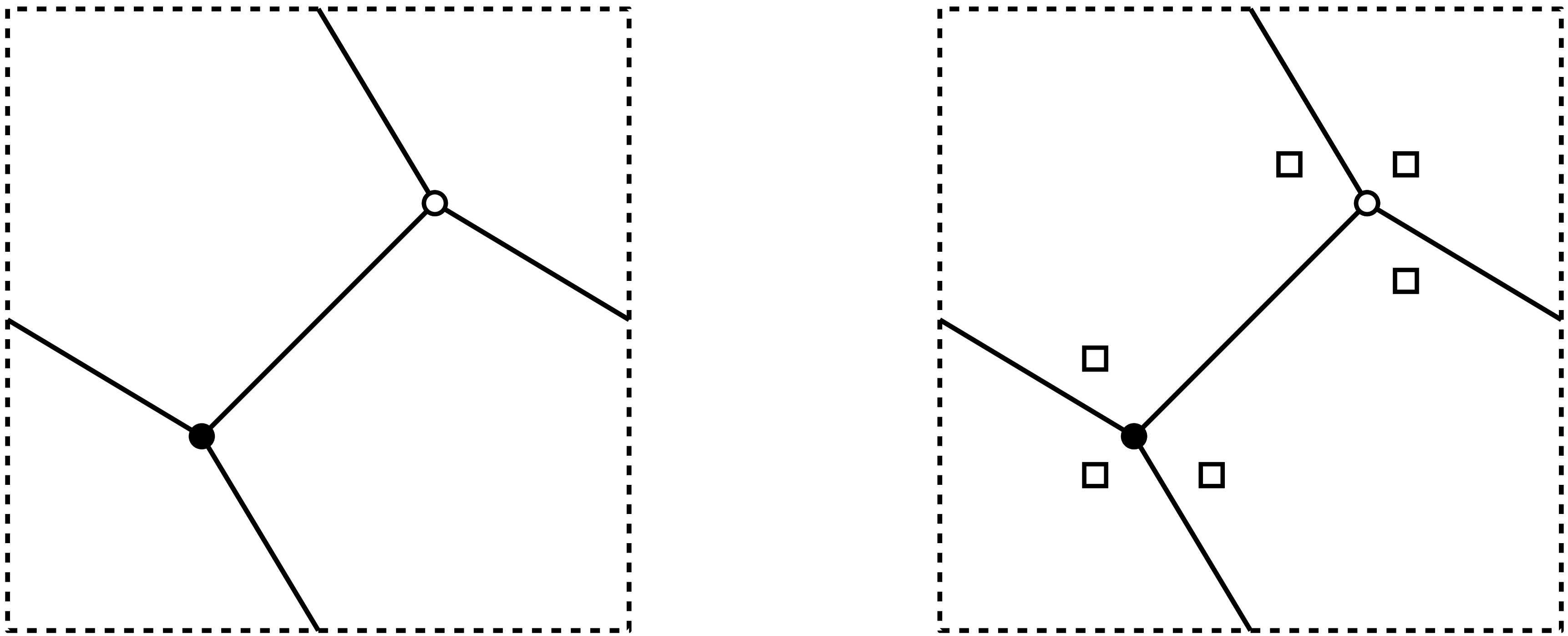
Fisher graphs

There is a correspondence between the Ising model on a given graph and the dimer model on the graph obtained by Fisher decoration.

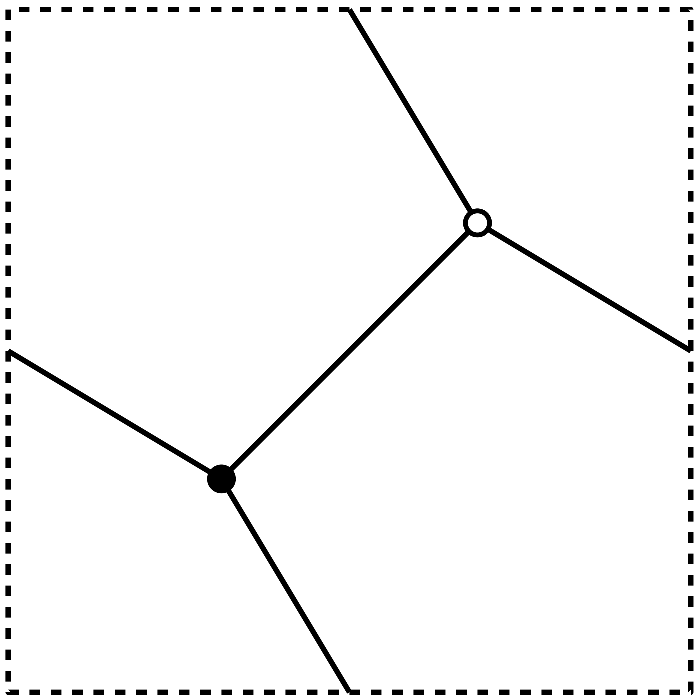


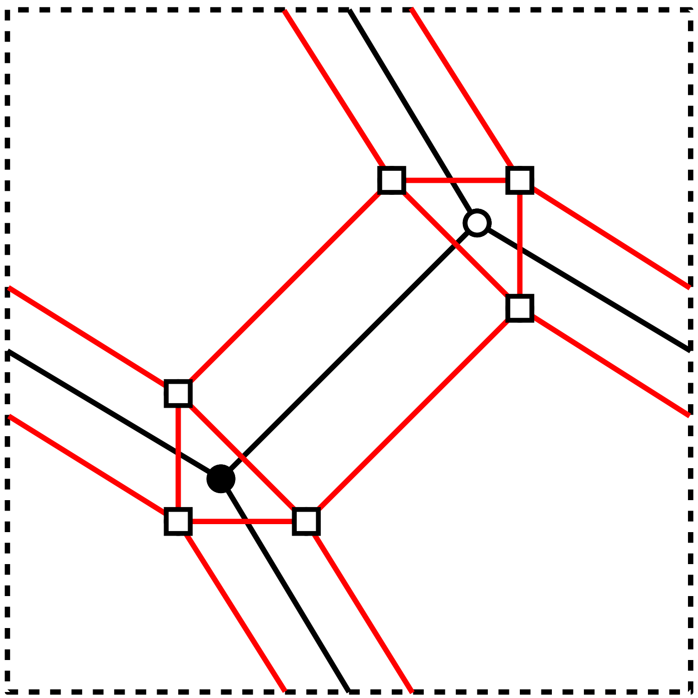
Cimasoni and Duminil-Copin (2013) and Li (2014) characterized the spectral curves arising from the dimer model on Fisher graphs when its edge weights are ≥ 1. It is an open problem to extend this result to any positive real weights.

Corner graphs

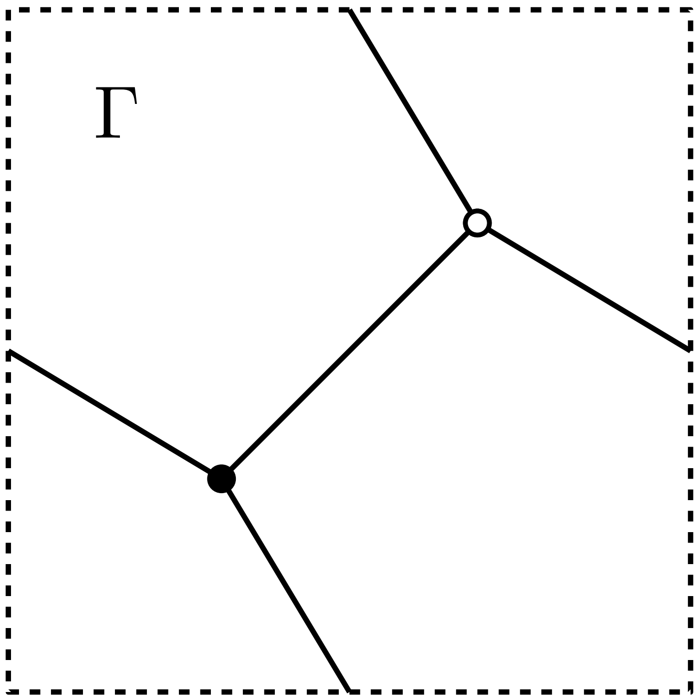


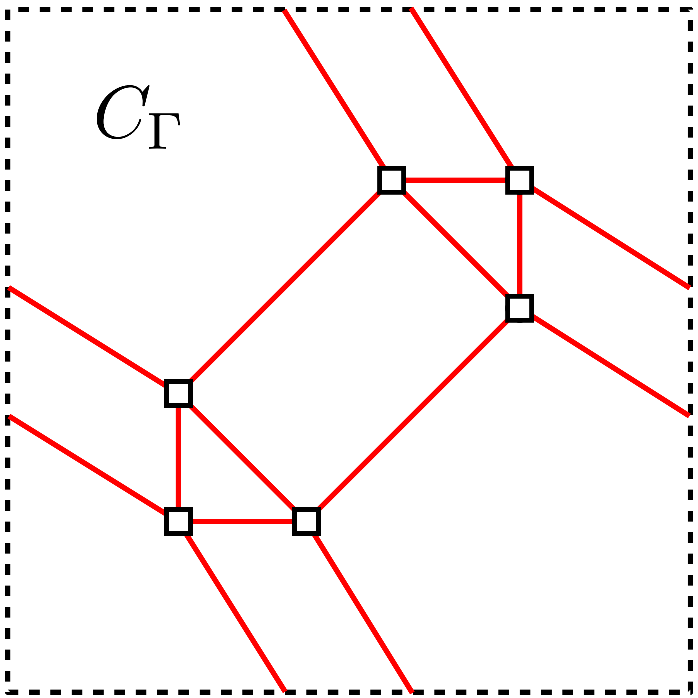
Corner graphs





Corner graphs

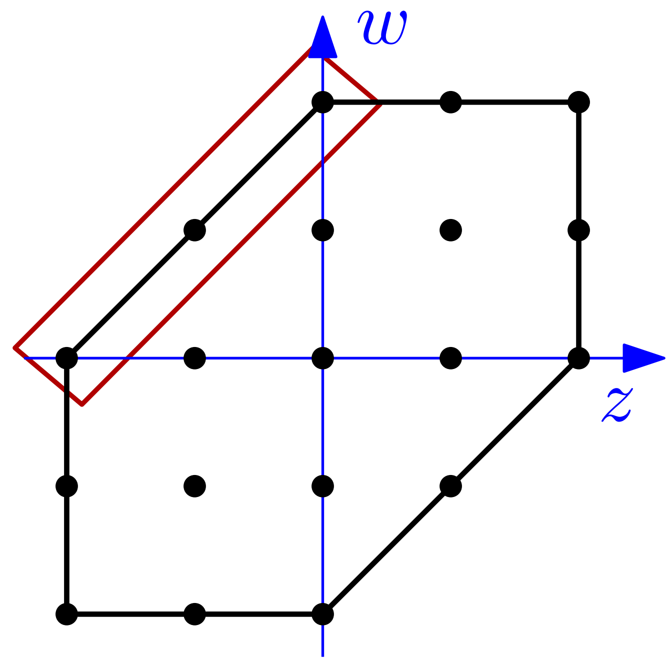




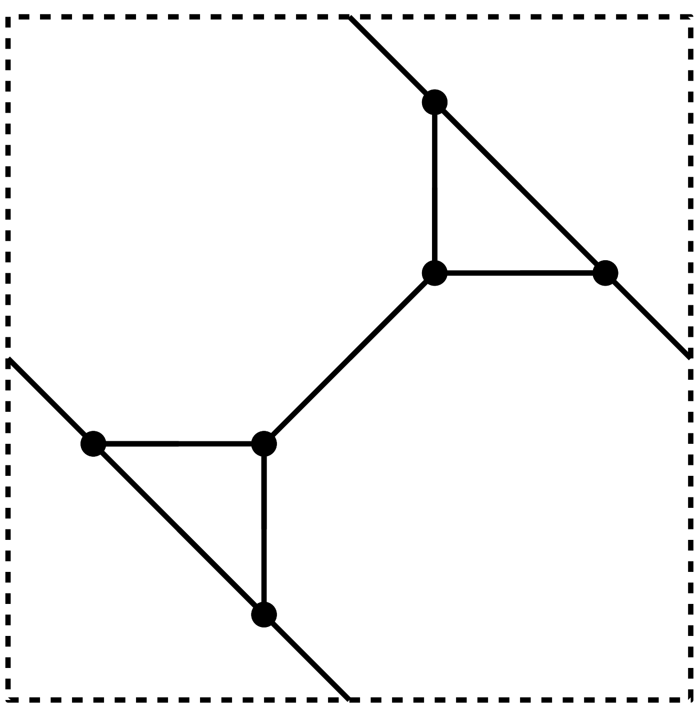
Theorem 2 [B. 2025]
Let Γ be a minimal bipartite graph with vertices that have degrees of the same parity. The edge vectors of the Newton polygon of C_Γ are obtained by homology classes of its zig-zag paths.

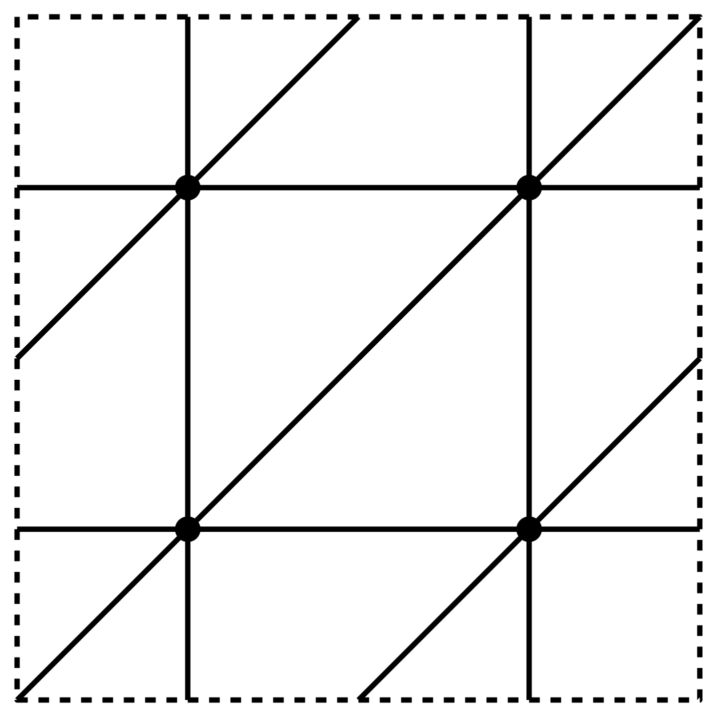
Marginal polynomials

$$P(z,w) = a_{0,0} + \frac{a_{-1,0}}{z} + a_{0,1}w + \boxed{\frac{a_{-2,0}}{z^2} + \frac{a_{-1,1}w}{z} + a_{0,2}w^2} + \dots$$

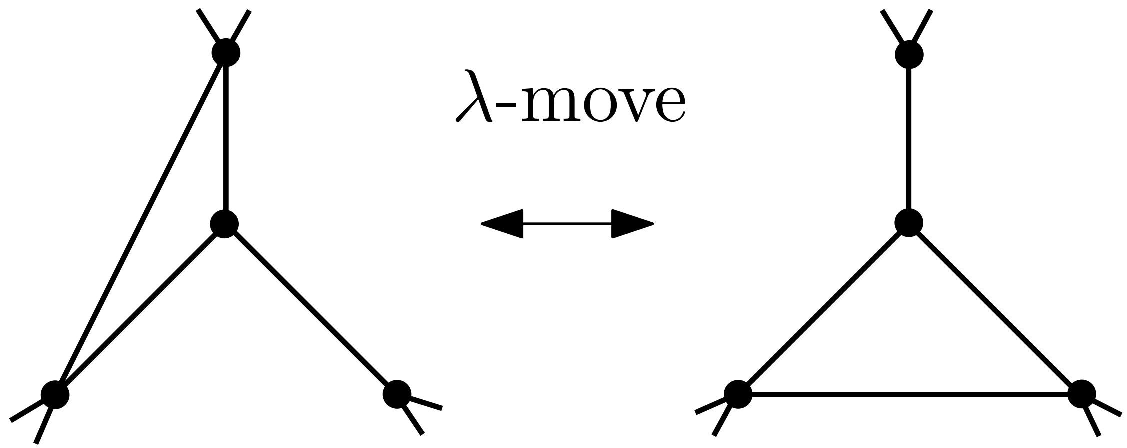


Theorem 3 [B. 2025]
Let N be the Newton polygon associated to the dimer model on the Fisher graph of the hexagonal lattice or a triangular lattice of arbitrary fundamental domain. Then the marginal polynomials of N are real-rooted and there is an explicit formula for their roots in factorized form.





Local move

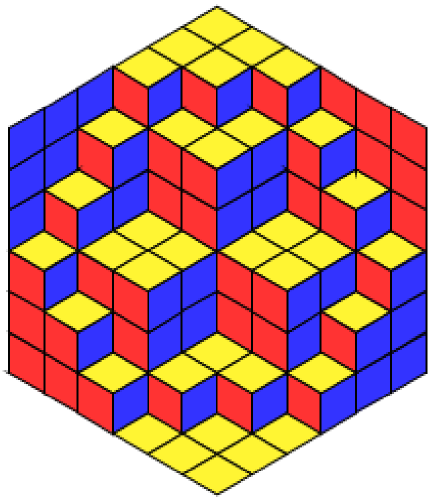


Proposition [B. 2025]
A λ-move preserves the dimer partition function.

Goncharov and Kenyon (2013) showed that any two minimal bipartite graphs with the same Newton polygon are related by two types of local moves.
Is there such an analogue in the non-bipartite case?

Dimers and ASMs

- We have seen that domino tilings of Aztec diamond, as well as rhombus tilings are related to ASMs.



- ASMs can be also related to certain dimer models of non-planar graphs.

- Is there a way to relate ASMs with dimers on non-bipartite graphs?

THANK YOU

SO MUCH !